Northwestern University Department of Physics and Astronomy

PHYSICS 428-1 QUANTUM FIELD THEORY I

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Course Webpage: `http://www.hep.anl.gov/ian/teaching/QFT/QFT_Fall08.html`

ASSIGNMENT #6

Due at 3:30 PM, November 7th

(Two pages and four problems.)

Reading Assignments:
Sections 3.6 of Peskin and Schroeder.

Problem 1
Do Problem 3.2 in Peskin and Schroeder.

Problem 2
We are going to consider the Lorentz transformation property of the Dirac bilinear $\Psi^{\mu\nu} = \bar{\psi}\sigma^{\mu\nu}\psi$. Because $\Psi^{\mu\nu}$ is anti-symmetric, there are six independent components in it. On the other hand, an irreducible representation (j_1, j_2) of $SO(1,3)$ has $(2j_1+1)\times(2j_2+1)$ independent components. (This is called the dimension of the representation.)
(a) Write down all the irreducible representations of $SO(1,3)$ that are six-dimensional. Show that $\Psi^{\mu\nu}$ can be none of them. (Hint: What is the parity of $\Psi^{\mu\nu}$?)
(b) Next we need to consider reducible representations of the form $(j_1, j_2)\oplus(j_1', j_2')$. By counting the dimensionality show that there is a unique reducible representation in which $\Psi^{\mu\nu}$ could possibly fit.
(c) Recall in classical electromagnetism we define the dual of the field strength as $\tilde{F}_{\mu\nu} = (1/2)\epsilon^{\mu\nu\delta\rho}F_{\delta\rho}$. Let's define

$$\Psi^{\mu\nu}_{\pm} = \Psi^{\mu\nu} \pm i\tilde{\Psi}^{\mu\nu}.$$

Then we can decompose $\Psi^{\mu\nu} = \Psi^{\mu\nu}_+ + \Psi^{\mu\nu}_-$. Show that $\Psi^{\mu\nu}_\pm$ are *self-dual* tensors which satisfy

$$\tilde{\Psi}^{\mu\nu}_{\pm} = \mp i\tilde{\Psi}^{\mu\nu}_{\pm}.$$

Again use the dimensionality counting to show that $\Psi^{\mu\nu}\pm$ would each fits into a unique irreducible representation of the Lorentz group.
(d) Now prove your answers in (b) and (c) by explicitly working out the transformation property of $\Psi^{\mu\nu}$ under the Lorentz group. (Hint: as always, to see how an object O transforms under the Lorentz group, you compute the commutator of O with the generators of $SO(1,3)$.)

Problem 3
(a) Show that the Dirac Lagrangian is invariant under space-time translations: $x^\mu \to x^\mu + a^\mu$.
(b) Derive the energy-momentum tensor of the Dirac field – the Noether current corresponding to translations.
(c) Construct the expression for the conserved physical momentum $\mathbf{P}$ in the quantum theory (that is, express it in terms of raising and lowering operators, $b^s_{\mathbf{p}}, c^s_{\mathbf{p}}$, etc.). Make sure you

discard infinite irrelevant additive constants and argue why you are doing it.
(d) Consider a one particle state,

$$|\mathbf{p}, s\rangle = \sqrt{2E_{\mathbf{p}}}\, b_{\mathbf{p}}^{s\dagger}|0\rangle,$$

and show that this is indeed a state of definite momentum.

Problem 4
(a) Derive Eqs. (3.114) and (3.115) in Peskin and Schroeder. Use them to show that $\psi_a(x)$ and $\bar{\psi}_b(y)$ anti-commute at space-like separations. That is, show that $\{\psi_a(x), \bar{\psi}_b(y)\} = 0$ for $(x-y)^2 < 0$.
(b) Consider two operators, $\mathcal{O}_1(x) = \bar{\psi}_a(x) A_{ab} \psi_b(x)$ and $\mathcal{O}_2(x) = \bar{\psi}_a(x) B_{ab} \psi_b(x)$, where A and B are some matrices. (All operators corresponding to physically observable quantities of a fermion field have this generic form – consider for example the energy-momentum tensor and the Noether current encountered earlier in this homework.) Prove that $\mathcal{O}_1$ and $\mathcal{O}_2$ *commute* at space-like separations:

$$[\mathcal{O}_1(x), \mathcal{O}_2(y)] \;=\; 0 \quad \text{at} \quad (x-y)^2 < 0,$$

as required by causality.